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An Alternative Decomposition of the Strain Gradient Tensor

An alternative decomposition of the strain gradient tensor is proposed in this paper in order to ensure that the deviatoric strain gradient vanishes for an arbitrary volumetric strain field, which is consistent with the physical picture of plastic deformation. The theory of mechanism-based strain gradient (MSG) plasticity is then modified accordingly based on this new decomposition. The numerical study of the crack-tip field based on the new theory shows that the crack tip in MSG plasticity has the square-root singularity, and the stress level is much higher than the HRR field in classical plasticity.
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1 Introduction

Fleck and Hutchinson [1] proposed a phenomenological theory of strain gradient plasticity in order to characterize the size dependence observed in the micron and submicron scale experiments ([2–9]). The strain gradient tensor $\eta_{ijk}=u_{k,ij}$ is decomposed into a volumetric part η^H_{ijk} and a deviatoric part η'_{ijk}, $\eta_{ijk}=\eta^H_{ijk}+\eta'_{ijk}$, where u_k is the displacement, and $\eta^H_{ijk}=1/4\,(\delta_{ik}\eta_{jpp}+\delta_{jk}\eta_{ipp})$ ([10]). Within the same theoretical framework ([1]), Gao, Huang and co-workers ([11,12]) developed the mechanism-based strain gradient (MSG) plasticity theory from the Taylor model in dislocation mechanics, and the theory agrees very well with the micro-indentation, microtorsion, and microbend experiments ([13,14]).

Hwang and Inoue [15] investigated the strain gradient effect for the following displacement field:

$$
\begin{aligned}
u_1&=A(x_1^2-x_2^2-x_3^2)+2Bx_1x_2+2Cx_1x_3,\\
u_2&=2Ax_1x_2+B(-x_1^2+x_2^2-x_3^2)+2Cx_2x_3, \qquad (1)\\
u_3&=2Ax_1x_3+2Bx_2x_3+C(-x_1^2-x_2^2+x_3^2),
\end{aligned}
$$

where A, B, and C are constants. It gives a pure volumetric strain field, $\varepsilon_{ij}=2(Ax_1+Bx_2+Cx_3)\delta_{ij}$, i.e., the deviatoric strain field ε'_{ij} vanishes. The strain gradient field, however, is not pure volumetric because the deviatoric strain gradient field does not vanish, $\eta'_{ijk}\neq 0$. It is quite puzzling that a pure volumetric strain field gives a deviatoric strain gradient field because the former implies no plastic deformation (since plastic deformation is always deviatoric) while the latter represents the plastic deformation associated with the geometrically necessary dislocations ([11]). It should be pointed out that the above puzzle between the volumetric strain field and deviatoric strain gradient field does not apply to the flow theories of strain gradient plasticity ([1,16–19]) because of the clear distinction between the plastic strain and the total strain. It does not affect the deformation theory of MSG plasticity ([11,12]) either since the theory assumes material incompressibility.

An alternative decomposition of the strain gradient tensor is proposed in this study,

$$\eta_{ijk}=\bar{\eta}^H_{ijk}+\bar{\eta}'_{ijk}, \qquad (2)$$

which gives a vanishing deviatoric part $\bar{\eta}'_{ijk}$ for an arbitrary volumetric strain field. The theory of MSG plasticity ([11,12]) is then generated accordingly to include the elastic deformation. Finally, we study the crack-tip field with the elastic-plastic theory of MSG plasticity, and show that the stress field around the crack tip has the square-root singularity.

2 Decomposition of the Strain Gradient Tensor

Because the strain gradient tensor can be expressed in terms of the strain, $\eta_{ijk}=\varepsilon_{ik,j}+\varepsilon_{jk,i}-\varepsilon_{ij,k}$, a natural way to define the deviatoric strain gradient is to replace the strain by its deviatoric part $\varepsilon'_{ij}(=\varepsilon_{ij}-1/3\varepsilon_{kk}\delta_{ij})$ in the above relation, i.e.,

$$\bar{\eta}'_{ijk}=\varepsilon'_{ik,j}+\varepsilon'_{jk,i}-\varepsilon'_{ij,k}, \qquad (3)$$

which clearly vanishes for a purely volumetric strain field [e.g., (1)]. The corresponding volumetric part of the strain gradient tensor becomes

$$\bar{\eta}^H_{ijk}=\eta_{ijk}-\bar{\eta}'_{ijk}=\frac{1}{3}\varepsilon_{pp,j}\delta_{ik}+\frac{1}{3}\varepsilon_{pp,i}\delta_{jk}-\frac{1}{3}\varepsilon_{pp,k}\delta_{ij}. \qquad (4)$$

The above decomposition is different from the existing strain gradient theories ([1,11,12]), and it ensures that the deviatoric and volumetric part of the strain gradient field result from the deviatoric and volumetric strain fields, respectively.

The higher-order stress, which is the work conjugate of the strain gradient tensor, is decomposition differently, $\tau_{ijk}=\bar{\tau}^H_{ijk}+\bar{\tau}'_{ijk}$, such that the virtual work done by the higher-order stress can be separated into the hydrostatic and deviatoric parts

$$\delta w=\tau_{ijk}\delta\eta_{ijk}=\bar{\tau}^H_{ijk}\delta\bar{\eta}^H_{ijk}+\bar{\tau}'_{ijk}\delta\bar{\eta}'_{ijk}. \qquad (5)$$

This requires the cross terms $\bar{\tau}^H_{ijk}\delta\bar{\eta}'_{ijk}$ and $\bar{\tau}'_{ijk}\delta\bar{\eta}^H_{ijk}$ to vanish, which gives the unique decomposition of the higher-order stress as

$$\bar{\tau}'_{ijk}=\tau_{ijk}-\bar{\tau}^H_{ijk}, \qquad (6)$$

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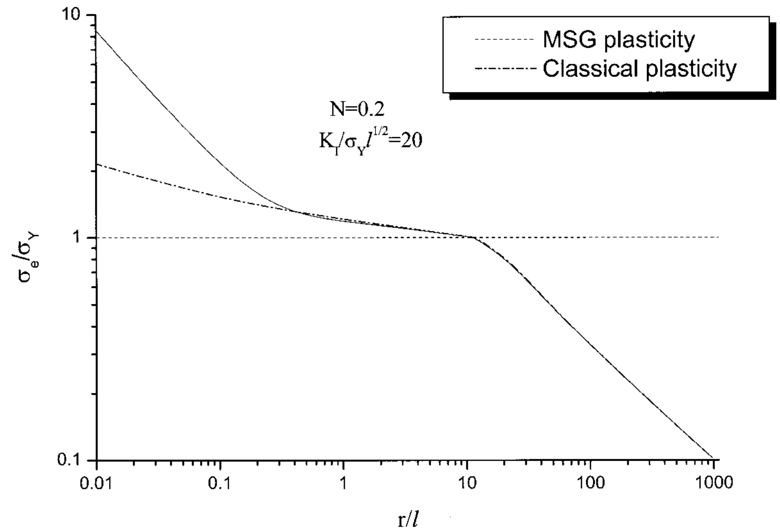

Fig. 1 The effective stress σ_e normalized by the uniaxial yield stress σ_Y versus the normalized distance to the crack tip, r/l, ahead of the crack tip, where l is the intrinsic material length in strain gradient plasticity; the plastic work hardening exponent $N=0.2$, Poisson's ratio $\nu=0.3$, the ratio of yield stress to elastic modulus $\sigma_Y/E=0.2$ percent, and the remotely applied elastic stress intensity factor $K_I/\sigma_Y l^{1/2}=20$

$$\overline{\tau}^H_{ijk}=\frac{1}{3}\,\delta_{ik}\left(\tau_{jpp}-\frac{1}{2}\,\tau_{ppj}\right)+\frac{1}{3}\,\delta_{jk}\left(\tau_{ipp}-\frac{1}{2}\,\tau_{ppi}\right). \qquad (7)$$

Unlike other strain gradient theories ([1,11,12]), the decomposition of the higher-order stress τ_{ijk} is different from that of the strain gradient η_{ijk}.

3 The Elastic-Plastic Theory of Mechanism-Based Strain Gradient Plasticity

Let $\sigma=\sigma_{\mathrm{ref}}f(\varepsilon)$ be the uniaxial stress-strain relation, and σ_{ref} be a reference stress in uniaxial tension. The flow stress σ in MSG plasticity is established from Taylor model in dislocation mechanics as ([13])

$$\sigma=\sqrt{\sigma_{\mathrm{ref}}^2 f^2(\varepsilon)+18\alpha^2\mu^2 b\,\eta}=\sigma_{\mathrm{ref}}\sqrt{f^2(\varepsilon)+l\,\eta}, \qquad (8)$$

where $\varepsilon=\sqrt{2/3\varepsilon'_{ij}\varepsilon'_{ij}}$ is the effective strain, μ the shear modulus, b the Burgers vector, α (0.1~0.5) an empirical material constant in the Taylor dislocation model, and the effective strain gradient η is determined by three dislocation models ([11]) for an incompressible solid as $\eta=1/2\sqrt{\eta'_{ijk}\eta'_{ijk}}$. Here the deviatoric strain gradient tensor η'_{ijk} is the same as $\overline{\eta}'_{ijk}$ in (3) for an incompressible solid, therefore a natural generalization of η for an elastic-plastic (compressible) solid is

$$\eta=\frac{1}{2}\sqrt{\overline{\eta}'_{ijk}\overline{\eta}'_{ijk}}. \qquad (9)$$

The parameter l in (8) is the intrinsic material length in strain gradient plasticity given by

$$l=18\alpha^2\left(\frac{\mu}{\sigma_{\mathrm{ref}}}\right)^2 b, \qquad (10)$$

which is on the order of a few microns.

Following the same multiscale approach ([11]), we have established the constitutive law for the elastic-plastic theory of MSG plasticity based on the alternative decomposition of the strain gradient tensor in (2)–(4).

$$\sigma_{ij}=K\varepsilon_{kk}\delta_{ij}+\frac{2\sigma}{3\varepsilon}\,\varepsilon'_{ij}, \qquad (11)$$

$$\tau_{ijk}=l_\varepsilon^2\left[\frac{K}{24}(\delta_{ik}\eta_{jpp}+\delta_{jk}\eta_{ipp})+\frac{\sigma}{\varepsilon}(\Lambda_{ijk}-\Pi_{ijk})+\frac{\sigma_{\mathrm{ref}}^2 f(\varepsilon)f'(\varepsilon)}{\sigma}\Pi_{ijk}\right], \qquad (12)$$

where K is the elastic bulk modulus; σ is the flow stress in (8);

$$\Lambda_{ijk}=\frac{1}{72}(2\,\overline{\eta}'_{ijk}+\overline{\eta}'_{kji}+\overline{\eta}'_{kij}),\quad \Pi_{ijk}=\frac{\varepsilon'_{mn}}{54\varepsilon^2}(\varepsilon'_{ik}\overline{\eta}'_{jmn}+\varepsilon'_{jk}\overline{\eta}'_{imn}); \qquad (13)$$

$l_\varepsilon=10(\mu/\sigma_Y)b$, and σ_Y is the initial yield stress in uniaxial tension.

4 Crack-Tip Singularity in MSG Plasticity

We use the finite element method for the elastic-plastic theory of MSG plasticity to investigate the mode I crack-tip field and crack-tip singularity. A semi-infinite crack in an infinite elastic-plastic solid remains traction-free on the crack face. The elastic K field is imposed on the remote boundary. The plastic work-hardening exponent $N=0.2$, the ratio of yield stress to Young's Modulus $\sigma_Y/E=0.2$ percent and Poisson's ratio $\nu=0.3$. Details of the numerical analysis are omitted in this paper.

Figure 1 shows the normalized Von Mises effective stress, σ_e/σ_Y, versus the nondimensional distance to the crack tip, r/l, ahead of the crack tip, where σ_Y is the yield stress and l is the intrinsic material length in strain gradient plasticity. The results are presented for both the elastic-plastic theory of MSG plasticity and the classical theory of plasticity (i.e., without strain gradient effects). The remote applied stress intensity factor is $K_I/\sigma_Y l^{1/2}=20$. The horizontal line of $\sigma_e/\sigma_Y=1$ separates the elastic and plastic zones. Outside the plastic zone, both curves emerge to the same straight lines with the slope of $-1/2$, corresponding to the elastic K field with the square-root singularity. Within the plastic zone, the two curves are also essentially the same at a distance larger than 0.4l to the crack tip. Within 0.4l to the crack tip, MSG plasticity theory predicts significantly larger stresses than their counterparts in classical plasticity. Moreover, classical plasticity theory gives a straight line with the slope of $-N/(N+1)$, corresponding to the HRR field ([20,21]), while MSG plasticity theory gives another straight line of slope $-1/2$, corresponding to the square-root singularity. In other words, stresses have the square-root singularity around a crack tip in MSG plasticity.

5 Concluding Remarks

We have proposed an alternative decomposition of the strain gradient tensor in order to ensure that the deviatoric strain gradient tensor vanishes for an arbitrary volumetric strain field $\varepsilon_{ij} = \varepsilon(\mathbf{x})\delta_{ij}$. This is consistent with the physical picture of plastic deformation since a pure volumetric strain field does not correspond to any plastic deformation (and therefore no dislocation activities) such that the deviatoric strain gradient should vanish since the latter is related to the density of geometrically necessary dislocations. We have modified the theory of mechanism-based strain gradient (MSG) plasticity ([11–13]) according to this new decomposition of the strain gradient tensor. We have then used the finite element method to investigate the crack-tip field in MSG plasticity, and have established that the crack tip has the square-root singularity. Within a distance on the order of microns to crack tip, the stress level predicted by MSG plasticity is significantly higher than the HRR field for classical plasticity.

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